Stochastic Algorithms for Solving a Multiperiod Quantile-Based Portfolio Optimization Problem

Anthony Coache   François Watier

Department of Mathematics
Université du Québec à Montréal

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Coherence

Coherent risk measure

Consider two random variables X and Y. A risk measure ρ is said to be **coherent** (Delbaen, 2002) if

- (Addition of capital) For all $a \in \mathbb{R}$, $\rho(X + a) = \rho(X) - a$;
- (Diversification principle) $\rho(X + Y) \leq \rho(X) + \rho(Y)$;
- (Proportional risk) For all $t \geq 0$, $\rho(tX) = t\rho(X)$;
- (Sure gain) If $X \geq 0$, then $\rho(X) \leq 0$.

Variance and value-at-risk (Artzner et al., 1999) are not coherent risk measures.

Quantile-based risk measures

Many coherent risk measures are proposed in the literature, such as the conditional value-at-risk (Rockafellar and Uryasev, 2002) and expected-shortfall (Acerbi and Tasche, 2002).

We focus on the coherent risk measure based on the α-quantile q_α

$$\begin{aligned} \rho_\alpha(X) &= \frac{-1}{\alpha}\left(\mathrm{E}\left[X\mathbb{1}(X \le q_\alpha(X))\right] + q_\alpha(X)\left(\alpha - \mathrm{P}\left[X \le q_\alpha(X)\right]\right)\right) \\ &= \frac{-1}{\alpha}\int_0^\alpha q_u(X)\mathrm{d}u. \end{aligned}$$

If the random variable X is continuous, then ρ_α is equivalent to the expected-shortfall.

Multiperiod portfolio optimization problem

- Coherent risk measures
- Multiperiod portfolio optimization problem
- Stochastic search algorithms
- Performance of search methods

Illustration of the problem

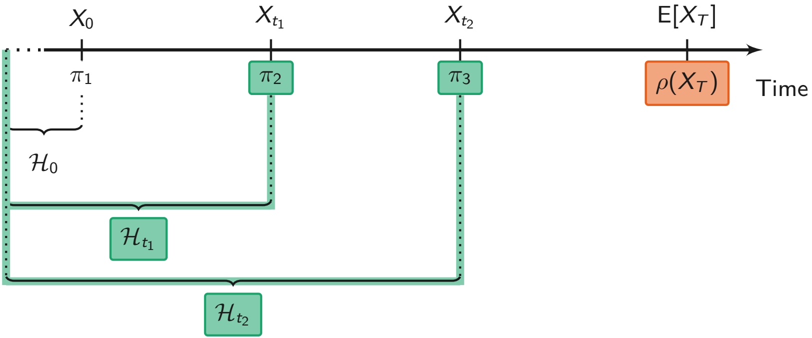
Invest our initial wealth and control the expected terminal wealth;

Illustration of the problem

Invest our initial wealth and control the expected terminal wealth;

Minimize a coherent risk measure;

Illustration of the problem



Invest our initial wealth and control the expected terminal wealth;

Minimize a coherent risk measure;

Adjust our strategy according to market fluctuations.

Why consider a binomial model?

- Discrete-time models are easier to handle.
- We can condition on specific wealth values since random variables only take a finite number of values.
- Binomial models can be generalized (e.g. trinomial model, multinomial model, etc.).

Under some assumptions, the binomial model provides a discrete-time approximation of the Black-Scholes model (Kim et al., 2016).

Parameters of the problem

We suppose that the interest rate of the riskless asset $r \equiv 0$ w.l.o.g., which means all rates of return are discounted.

- Initial wealth: X_0
- Wealth at the end of period i: X_i
- Expected terminal wealth: X^*
- Wealth amount invested in the risky asset at the beginning of period i: π_i
- Rate of return of the risky asset at the end of the period i: R_i
    - Probability of a high-reward rate of return: p
    - High-reward and low-reward rates of returns: U and L
- Threshold for the risk measure: α
- Number of periods: n

Portfolio optimization problem with a binomial model

Definition of the problem

$$\min_{\boldsymbol{\pi}} \ \rho_\alpha(X_n) \quad \text{s.t.} \quad \mathrm{E}[X_n] = X^*.$$

- Self-financing constraint:

$$X_n = X_{n-1}(1+r) + \pi_n(R_n - r) = X_0 + \sum_{i=1}^{n} \pi_i R_i.$$

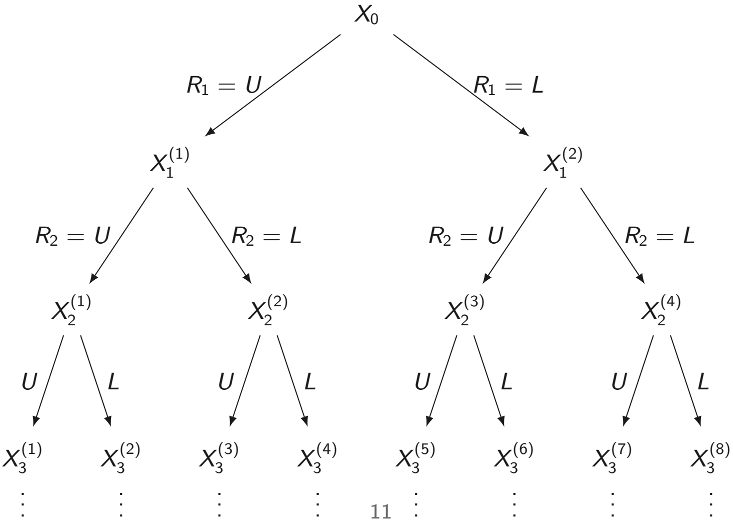
- One risky asset with independent rates of return over each period:

$$R_i = \begin{cases} U & \text{with probability } p \\ L & \text{with probability } 1-p \end{cases}, \quad \forall i = 1, \dots, n.$$

- U, L and p are chosen such that $U > 0$, $L < 0$ and $\mathrm{E}[R_i] > 0$.

Portfolio optimization problem with a binomial model

π_1 is a constant and $\pi_i,\ i = 2, \ldots, n$ are random variables. Thus there are 2^n possible terminal wealth values:



Where to look for the global minimum?

Proposition

The following risk measure is a **convex** function :

$$\rho_\alpha(X) = \frac{-1}{\alpha}\left(\mathrm{E}\left[X\mathbb{1}(X \leq q_\alpha(X))\right] + q_\alpha(X)\left(\alpha - \mathrm{P}\left[X \leq q_\alpha(X)\right]\right)\right)$$
$$= \frac{-1}{\alpha}\int_0^\alpha q_u(X)\mathrm{d}u.$$

The solution of this convex optimization problem is on the boundaries. The global minimum is necessarily obtain when the random variable X_n takes two unique values.

Where to look for the global minimum?

- Partition possible terminal wealth values into two groups;
- Solve both linear systems such that every wealth values in a group is equal;
- Compute ρ_α with these terminal wealth values.

Number of different combinations

$$\sum_{k=1}^{2^{n-1}-1} \binom{2^n}{k} + \frac{1}{2}\binom{2^n}{2^{n-1}} = 2^{2^n-1} - 1.$$

Stochastic search algorithms are essential to avoid enumerating each and every possible partitioning.

$$(2^{2^6-1} - 1 \approx 9.2 \times 10^{18} \text{ partitions...})$$

Stochastic search algorithms

- Coherent risk measures
- Multiperiod portfolio optimization problem
- **Stochastic search algorithms**
- Performance of search methods

Discrete uniform search algorithm (without replacement)

Input: Initialization of parameters ;
Initialize the number of iterations ;
Generate a discrete uniform without replacement sequence of integers;
Set the minimum risk measure to infinity ;
for *ii = 1* **to** *the number of iterations* **do**
    Select the *ii*-th partitioning ;
    Compute its associated risk measure ;
    if *it improves the minimum risk measure* **then**
        Update the minimum risk measure found ;
Output: The minimum risk measure and its associated partitioning ;

Discrete uniform search algorithm (without replacement)

- This algorithm is efficient on average to find the global minimum, indeed

$$\text{E[Nbr. of iterations to find global minimum]} = 2^{2^n-2}.$$

- It cannot visit the same combination twice.

- A big amount of memory space is required when the number of periods n grows. Its realization with $n \geq 6$ periods is almost impracticable on a single computer.

We then propose a stochastic algorithm that takes advantage of the structure of the problem and the binomial model.

Markovian change-when-improve algorithm

```
Input: Initialization of the number of iterations and parameters ;
Select randomly a partitioning ;
Compute its associated risk measure ;
Create a memory variable ;
for ii = 2 to the number of iterations do
    Select a label and remove it from the memory variable ;
    Change partitioning by switching group this label ;
    Compute the new associated risk measure ;
    if it improves the minimum risk measure then
        Update the minimum risk measure found ;
    if it improves the risk measure compared to last iteration then
        Update partitioning and memory variable ;
    else if the memory variable is empty then
        Reinitialize partitioning and the memory variable ;
        Compute its associated risk measure ;
Output: The minimum risk measure and its associated partitioning ;
```

Markovian change-when-improve algorithm

- It takes advantage of the structure of the problem.
- The Markovian change-when-improve search keeps track of at most the last 2^n combinations seen, which is less problematic when n grows.
- This algorithm changes partitioning only when there is an improvement of the cost function.

Next slides illustrate the different steps and variables states of the Markovian change-when-improve algorithm for a portfolio optimization problem with $n = 3$ periods.

Markovian change-when-improve - Illustration

P1 (1st partition)	[1 3]
P2 (2nd partition)	[2 4 5 6 7 8]
P1Temp	
P2Temp	
FctTemp (ρ_α)	-0.2471
Memory	[1 2 3 4 5 6 7 8]

FctLast (ρ_α) -0.2471
FctMin (ρ_α) -0.2471 CombinMin [1 3]
Global minimum -0.8595

Initialize combinations

Markovian change-when-improve - Illustration

P1 (1st partition)	[1 3]
P2 (2nd partition)	[2 4 5 6 7 8]
P1Temp	[1]
P2Temp	[2 3 4 5 6 7 8]
FctTemp (ρ_α)	-0.5281
Memory	[1 2 3 4 5 6 7 8]

FctLast (ρ_α) -0.2471
FctMin (ρ_α) -0.2471 CombinMin [1 3]
Global minimum -0.8595

Split, remove from memory and compute the risk measure

Markovian change-when-improve - Illustration

P1 (1st partition)	[1]
P2 (2nd partition)	[2 3 4 5 6 7 8]
P1Temp	[1]
P2Temp	[2 3 4 5 6 7 8]
FctTemp (ρ_α)	-0.5281
Memory	[1 2 3 4 5 6 7 8]

FctLast (ρ_α) -0.5281
FctMin (ρ_α) -0.5281
CombinMin [1]
Global minimum -0.8595

Update partitions and the global minimum

Markovian change-when-improve - Illustration

P1 (1st partition)	[1]
P2 (2nd partition)	[2 3 4 5 6 7 8]
P1Temp	[1 6]
P2Temp	[2 3 4 5 7 8]
FctTemp (ρ_α)	-0.2471
Memory	[1 2 3 4 5 6 7 8]

FctLast (ρ_α) -0.5281
FctMin (ρ_α) -0.5281 CombinMin [1]
Global minimum -0.8595

Split, remove from memory and compute the risk measure

Markovian change-when-improve - Illustration

P1 (1st partition)	[1]
P2 (2nd partition)	[2 3 4 5 6 7 8]
P1Temp	[1 6]
P2Temp	[2 3 4 5 7 8]
FctTemp (ρ_α)	-0.2471
Memory	[1 2 3 4 5 6 7 8]

FctLast (ρ_α) -0.5281
FctMin (ρ_α) -0.5281 CombinMin [1]
Global minimum -0.8595

Keep partitions since we do not improve the risk measure

Markovian change-when-improve - Illustration

P1 (1st partition)	[1]
P2 (2nd partition)	[2 3 4 5 6 7 8]
P1Temp	[1 2]
P2Temp	[3 4 5 6 7 8]
FctTemp (ρ_α)	-0.2471
Memory	[1 2 3 4 5 6 7 8]

FctLast (ρ_α) -0.5281
FctMin (ρ_α) -0.5281 CombinMin [1]
Global minimum -0.8595

Markovian change-when-improve - Illustration

P1 (1st partition)	[1]
P2 (2nd partition)	[2 3 4 5 6 7 8]
P1Temp	[1 7]
P2Temp	[2 3 4 5 6 8]
FctTemp (ρ_α)	-0.2471
Memory	[1 2 3 4 5 6 7 8]

FctLast (ρ_α) -0.5281
FctMin (ρ_α) -0.5281 CombinMin [1]
Global minimum -0.8595

Markovian change-when-improve - Illustration

P1 (1st partition)	[1]
P2 (2nd partition)	[2 3 4 5 6 7 8]
P1Temp	[1 3]
P2Temp	[2 4 5 6 7 8]
FctTemp (ρ_α)	-0.2471
Memory	[1 2 3 4 5 6 7 8]

FctLast (ρ_α) -0.5281
FctMin (ρ_α) -0.5281 CombinMin [1]
Global minimum -0.8595

Markovian change-when-improve - Illustration

P1 (1st partition)	[1]
P2 (2nd partition)	[2 3 4 5 6 7 8]
P1Temp	[1 8]
P2Temp	[2 3 4 5 6 7]
FctTemp (ρ_α)	-0.3600
Memory	[1 2 3 4 5 6 7 8]

FctLast (ρ_α) -0.5281
FctMin (ρ_α) -0.5281
CombinMin [1]
Global minimum -0.8595

Markovian change-when-improve - Illustration

P1 (1st partition)	[1]
P2 (2nd partition)	[2 3 4 5 6 7 8]
P1Temp	[1 4]
P2Temp	[2 3 5 6 7 8]
FctTemp (ρ_α)	-0.2471
Memory	[1 2 3 4 5 6 7 8]

FctLast (ρ_α) -0.5281
FctMin (ρ_α) -0.5281 CombinMin [1]
Global minimum -0.8595

Markovian change-when-improve - Illustration

P1 (1st partition)	[1]
P2 (2nd partition)	[2 3 4 5 6 7 8]
P1Temp	[1 5]
P2Temp	[2 3 4 6 7 8]
FctTemp (ρ_α)	-0.2471
Memory	[1 2 3 4 5 6 7 8]

FctLast (ρ_α) -0.5281
FctMin (ρ_α) -0.5281 CombinMin [1]
Global minimum -0.8595

Markovian change-when-improve - Illustration

P1 (1st partition)	[1 5 8]
P2 (2nd partition)	[2 3 4 6 7]
P1Temp	
P2Temp	
FctTemp (ρ_α)	
Memory	[1 2 3 4 5 6 7 8]

FctLast (ρ_α) -0.0036
FctMin (ρ_α) -0.5281 CombinMin [1]
Global minimum -0.8595

Markovian change-when-improve - Illustration

P1 (1st partition)	[1 5 8]
P2 (2nd partition)	[2 3 4 6 7]
P1Temp	[1 8]
P2Temp	[2 3 4 5 6 7]
FctTemp (ρ_α)	-0.3600
Memory	[1 2 3 4 5 6 7 8]

FctLast (ρ_α) -0.0036
FctMin (ρ_α) -0.5281 CombinMin [1]
Global minimum -0.8595

Markovian change-when-improve - Illustration

P1 (1st partition)	[1 8]
P2 (2nd partition)	[2 3 4 5 6 7]
P1Temp	[1 8]
P2Temp	[2 3 4 5 6 7]
FctTemp (ρ_α)	-0.3600
Memory	[1 2 3 4 5 6 7 8]

FctLast (ρ_α) -0.3600
FctMin (ρ_α) -0.5281 CombinMin [1]
Global minimum -0.8595

Markovian change-when-improve - Illustration

P1 (1st partition)	[1 8]
P2 (2nd partition)	[2 3 4 5 6 7]
P1Temp	[8]
P2Temp	[1 2 3 4 5 6 7]
FctTemp (ρ_α)	-0.8595
Memory	[1 2 3 4 5 6 7 8]

FctLast (ρ_α) -0.3600
FctMin (ρ_α) -0.5281 CombinMin [1]
Global minimum -0.8595

Markovian change-when-improve - Illustration

P1 (1st partition)	[8]
P2 (2nd partition)	[1 2 3 4 5 6 7]
P1Temp	[8]
P2Temp	[1 2 3 4 5 6 7]
FctTemp (ρ_α)	-0.8595
Memory	[1 2 3 4 5 6 7 8]

FctLast (ρ_α) -0.8595
FctMin (ρ_α) -0.8595 CombinMin [8]
Global minimum -0.8595

Global minimum is found

Performance of search methods

- Coherent risk measures
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Performance of search methods

Comparison of the uniform without replacement (Unif w/o) and Markovian change-when-improve (MCWI) algorithms

- Number of iterations to find the global minimum
- Minimum found after a fixed number of iterations

Here are the parameters of the problem with the same notation as specified earlier.

- Initial wealth: $X_0 = 1$
- Expected terminal wealth: $X^* = 6/5$
- Probability of a high-reward rate of return: $p = 0.75$
- High-reward rate of return: $U = 1$
- Low-reward rate of return: $L = -2$
- Threshold for the risk measure: $\alpha = 0.3$
- Number of periods: n

Number of iterations to find the global minimum

$$2^{2^2-1}-1=7$$

Uniform w/o replace search expectation :

4 iterations

MCWI search estimated expectation :

8.60 iterations

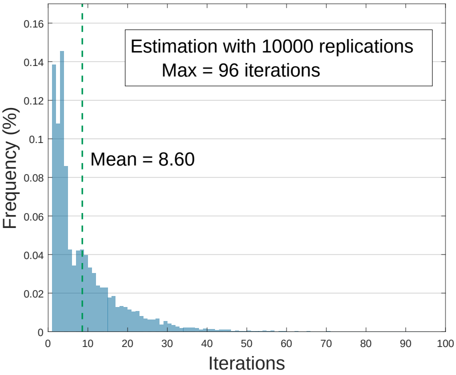


Figure: Distribution of the number of iterations to find the global minimum - MCWI 2 periods

Number of iterations to find the global minimum

$$2^{2^3-1}-1=127$$

Uniform w/o replace search expectation :

64 iterations

MCWI search estimated expectation :

25.79 iterations

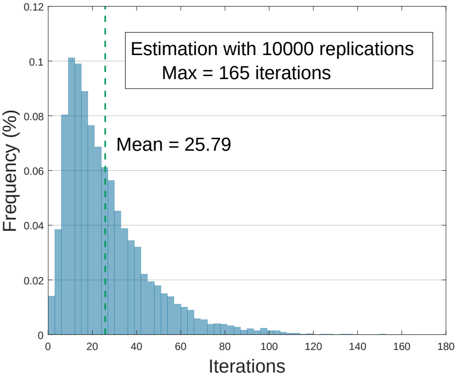


Figure: Distribution of the number of iterations to find the global minimum - MCWI 3 periods

Number of iterations to find the global minimum

$$2^{2^4-1} - 1 = 32\ 767$$

Uniform w/o replace search expectation :

16 384 iterations

MCWI search estimated expectation :

98.53 iterations

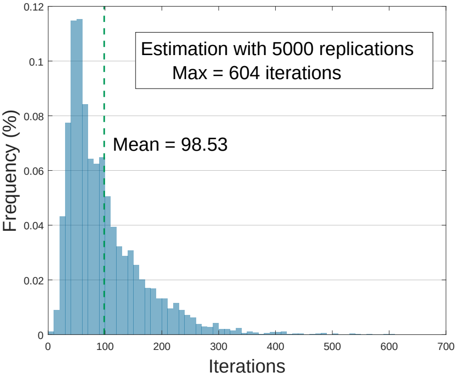


Figure: Distribution of the number of iterations to find the global minimum - MCWI 4 periods

Number of iterations to find the global minimum

$$2^{2^5-1} - 1 \approx 2\times10^9$$

Uniform w/o replace search expectation :

$\approx 1\times10^9$ iterations

MCWI search estimated expectation :

294.53 iterations

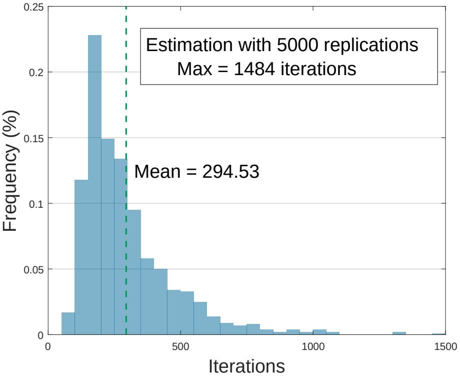


Figure: Distribution of the number of iterations to find the global minimum - MCWI 5 periods

Minimum found after a fixed number of iterations

Best of 4 iterations

Uniform w/o replace search replications that found the global minimum :

57.07%

MCWI search replications that found the global minimum:

48.22%

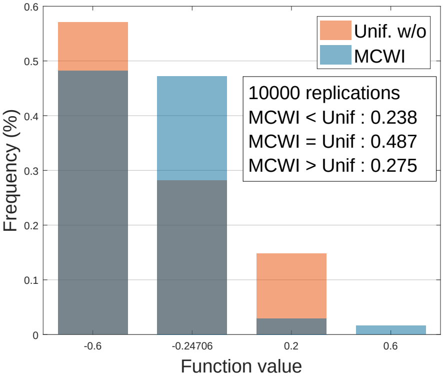
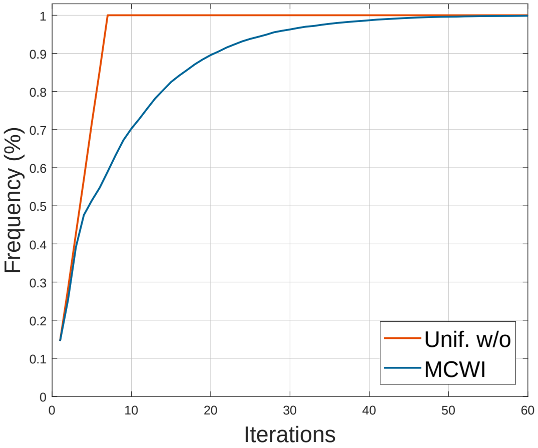


Figure: Distribution of the minimum values found by both algorithms - 2 periods

Uniform (w/o replace) vs. Markovian CWI

Figure: Proportion of 10000 replications that found the global minimum for both methods - 2 periods



Minimum found after a fixed number of iterations

Best of 65 iterations

Uniform w/o replace search replications that found the global minimum :

51.38%

MCWI search replications that found the global minimum:

95.23%

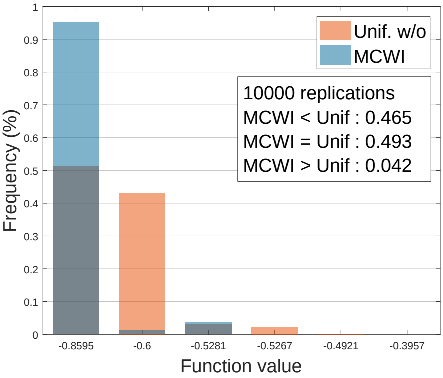
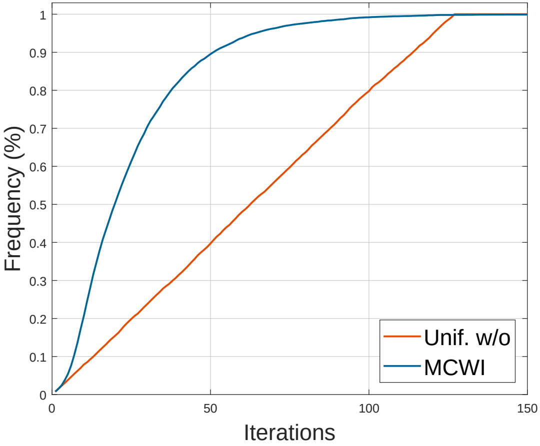


Figure: Distribution of the minimum values found by both algorithms - 3 periods

Uniform (w/o replace) vs. Markovian CWI

Figure: Proportion of 10000 replications that found the global minimum for both methods - 3 periods



Minimum found after a fixed number of iterations

Best of 150 iterations

Uniform w/o replace search replications that found the global minimum :

0.37%

MCWI search replications that found the global minimum:

82.64%

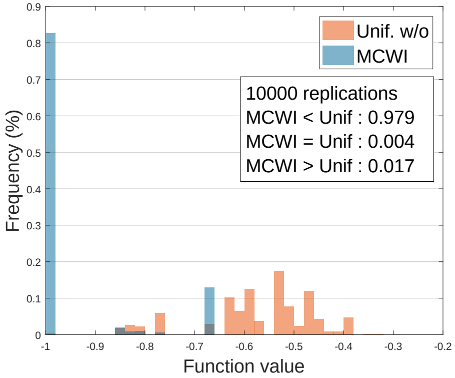
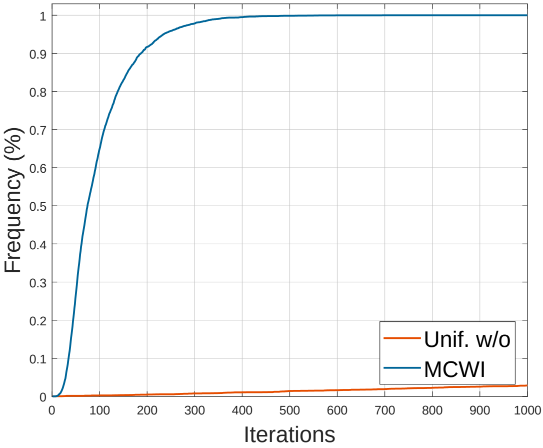


Figure: Distribution of the minimum values found by both algorithms - 4 periods

Uniform (w/o replace) vs. Markovian CWI

Figure: Proportion of 5000 replications that found the global minimum for both methods - 4 periods



Conclusion

- Stochastic algorithms provide efficient procedures to find optimal (or near-optimal) solutions to optimization problems.

- Using the structure of the problem improves significantly the efficiency of search algorithms in multiperiod portfolio optimization problems.

- It could provide some insights on the potential optimal strategy in the continuous case with models such as the Black-Scholes model.

Acknowledgments and references

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Thank you!

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